Macro II

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Introduction

- Today: reintroduce stochastic neoclassical growth model.
- Jumping off point for most modern macro models.
- Introduce solution techniques.
- Important to note “stochastic” here: our discussion over the past two weeks becomes important.

Model environment

- As an aside, since this is a straightforward model, let'ts talk about presentations.
- Make sure you tailor your talk to the audience.
- (Almost) Every macro presentation should have an environment slide.
- This details the following:
    1. preferences
    2. technology
    3. markets
- Next, what are the states and decisions of the agents?
- Then introduce the individual problem.
- We will go through each of these.

Consumer's Problem

- consumers in this economy maximize the expected value given by

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

- u is a bounded, continuous and strictly increasing utility function.
- $\beta \in (0,1)$ is a discount factor.
- subject to

$$y_t \geq c_t + [\ \underbrace{k_{t+1} - (1-\delta)k_t}_{=i_t}\ ]$$

- Note that lifetime utility is unknown: hence, agents require a market structure to insure themselves.

Resource Constraints

$$y_t \geq c_t + i_t$$
$$k_{t+1} = (1 - \delta)k_t + i_t$$

- An agent owns an amount $y_t \in \mathbb{R}_+ := [0, \infty)$ of consumption good at time t.
- Output can either be consumed or invested.
- When the good is invested, it is transformed one-for-one into capital.

Technology

$$y_{t+1} = f(\gamma_{t+1}, k_{t+1}) \; , \; \gamma_{t+1} \sim \Phi$$

- $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is the production function which is increasing and continuous in k and γ.
- Production is stochastic, in that it depends on a shock γ_{t+1} realized at the end of the current period t.
- Calibration

$$\begin{aligned} \gamma_{t+1} &= e^{\sigma \epsilon_{t+1}}, \epsilon_{t+1} \overset{i.i.d.}{\sim} \mathcal{N}(0,1), \; \sigma > 0 \\ f(k_{t+1}, \gamma_{t+1}) &= \gamma_{t+1} k_{t+1}^{\alpha} \\ \delta &= 1 \end{aligned}$$

Optimization

$$\max_{\{c_t,\ k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$s.t.\ y_t \geq c_t + k_{t+1}\ \forall t \qquad \text{(Resource Constraint)}$$

$$y_{t+1} = f(k_{t+1}, \gamma_{t+1}),\ \gamma_{t+1} \overset{i.i.d.}{\sim} \Phi\ \forall t \qquad \text{(Technology)}$$

$$c_t \geq 0\ k_{t+1} \geq 0\ \forall t \qquad \text{(Non-negativity Constraint)}$$

$$y_0 = \bar{y}_0\ given$$

Sequential Problem (SP)

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$s.t.\ y_{t+1} = f(k_{t+1}, \gamma_{t+1}),\ \gamma_{t+1} \overset{i.i.d.}{\sim} \Phi\ \forall t$$

$$y_t \geq c_t + k_{t+1} \forall\ t$$

$$y_0 = \bar{y}_0 \text{ given}$$

- Resource constraint holds with equality b/c $u' > 0$.
- y_t summarizes state of world at the start of each period.
- c_t is chosen by the agent each period after observing the state.

Functional Equation (FE)

(SP) is an infinite-dimensional optimization problem. Instead, find a time-invariant solution to functional equation:

$$
\begin{aligned}
v^*(y) &= \max_{c \in [0,y]} \left\{ u(c) + \beta \int v^*\Big(f(k', \gamma)\Big)\phi(d\gamma) \right\} \\
y &= c + k' \\
y' &= f(k', \gamma)
\end{aligned}
$$

Solution v^*, evaluated at $y = \bar{y}_0$, gives the value of the maximum in (SP).

Steady State

- Hard to characterize dynamics/solve model (find $c(t), k(t) \forall\ t$)
- Instead, characterize steady-state.
- $c = c' = c^*$, $k = k' = k^*$.
- pick $u(c) = ln(c)$, $f(k, \gamma) = e^{\sigma\epsilon} k^{\alpha}$ and $\epsilon \sim N(0,1), \sigma = 1$.
- then

$$\frac{1}{c} = \beta \mathbb{E}[(\alpha e^{\epsilon'} k'^{\alpha-1}) \frac{1}{c'}]$$

- In steady state:

$$\frac{1}{c^*} = \beta(\alpha \bar{\gamma} k^{*\alpha-1}) \frac{1}{c^*}$$

- We will solve for the *stochastic steady state*.
- i.e., the steady-state if the aggregate shock were at its mean.
- Does this differ from the steady-state?

Steady State

- This leaves us with capital:

$$1 = \beta(\alpha\bar{\gamma}k^{*\alpha-1})$$
$$k^* = (\frac{1}{\bar{\gamma}\alpha\beta})^{\frac{1}{\alpha-1}}$$
$$k^* = (\bar{\gamma}\alpha\beta)^{\frac{1}{1-\alpha}}$$

- Now consumption from the budget constraint:

$$c^* + k^* = \bar{\gamma}k^{*\alpha}$$
$$c^* = \bar{\gamma}k^{*\alpha} - k^*$$
$$c^* = \bar{\gamma}(\bar{\gamma}\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\bar{\gamma}\alpha\beta)^{\frac{1}{1-\alpha}}$$

Dynamics

- Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k'^{\alpha-1})c$$
$$k' = \bar{\gamma}k^{\alpha} - c$$

- We have two dynamic variables: c and k.
- The behavior of this system will depend on their dynamics.
- At steady-state:

$$1 = \frac{c'}{c} = \beta(\alpha\bar{\gamma}k^{\alpha-1})$$
$$1 = \frac{k'}{k} = \bar{\gamma}k^{\alpha-1} - \frac{c}{k}$$

- If both hold, we are in steady-state, if not, dynamics can vary.

Dynamics

- Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k^{\alpha-1})c$$
$$k' = \bar{\gamma}k^{\alpha} - c$$

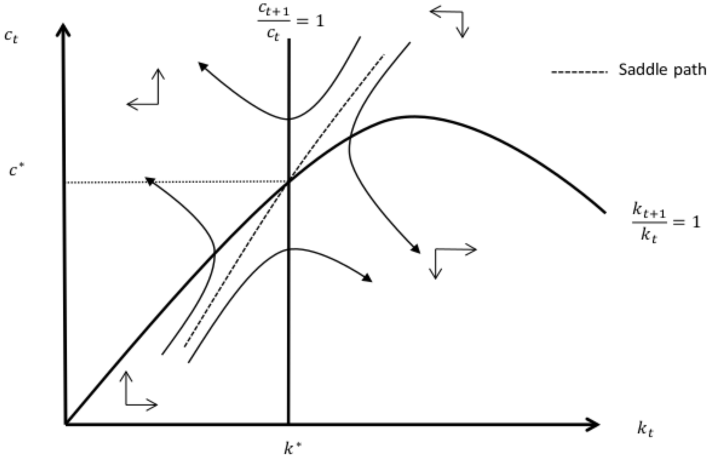
- Small c: second equation dictates that capital increases.
- Small k: first equation dictates that consumption increases.
- Reverse is true.

Phase Diagram

- Dynamics:

$$c' = \beta(\alpha\bar{\gamma}k'^{\alpha-1})c$$
$$k' = \bar{\gamma}k^{\alpha} - c$$

- From Eric Sim's notes:



Solving for dynamics

- Recall contraction mapping
- If T is a contraction mapping with modulus β, then
  1. there exists a unique fixed point v^*, and

$$\begin{aligned} v^* &= Tv^* \\ v^*(y) &= \max_{c\in[0,y]} \left\{ u(c) + \beta \int v^*\Big(f(k',\gamma)\Big)\phi(d\gamma) \right\} \\ y &= c + k' \\ y' &= f(k',\gamma) \end{aligned}$$

  2. for any v_0 and any $n \in \mathbb{N}$,

$$\rho(T^n v_0, v^*) \leq \beta^n \rho(v_0, v^*)$$

- Yields an is a policy function $\sigma^*(y) = argmax\{v(y)\}$
- We can apply this in two ways: analytically or computationally.

Computation

How can we implement Bellman operator on our computer?

$$Tw(y) := \max_{c \in [0,y]} \underbrace{\left\{ u(c) + \beta \underbrace{\int \underbrace{w\Big(f(k', \gamma)\Big)}_{\text{1. Approximation}} \phi(d\gamma)}_{\text{2. Integration}} \right\}}_{\text{3. Optimization}}$$

where w is a function that approximates v.

Approximation

- Approximate an analytically intractable real-valued function f with a computationally tractable function $\widehat{f}$
- given limited information about f.
- Divide the **approximation domain** of the function into **finite number of sub-intervals** and approximate the original function in each of the intervals.
  - The points on the domain which separate the intervals are called grid points.
  - We use the value of the function at each grid point to approximate the original function.
- Another way to think about it: sampling from domain of the function at n nodes. As $n \to \infty, \widehat{f} \to f$

Approximation II

In order to figure out Bellman operator, we need to approximate an analytically intractable real-valued function w.

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(k', \gamma)\Big)\phi(d\gamma) \right\}$$

- Interpolation
  1. Determine an approximation domain of w.
  2. Pick n (often evenly spaced) nodes, produces $n - 1$ intervals.
  3. Approximate the original function w in each of the resulting intervals using a polynomial.
- Grid search
  1. Determine an approximation domain of w.
  2. Pick n nodes, produces $n - 1$ intervals.
  3. Evaluate function at each node and pick maximum.

Integration

In order to figure out Bellman operator, we need to evaluate continuation value.

$$Tw(y) := \max_{c \in [0,y]} \left\{ u(c) + \beta \int w\Big(f(k', \gamma)\Big) \phi(d\gamma) \right\}$$

- One approach: Monte Carlo integration: Given a random sample of size n, $\{\gamma_i\}_{i=1}^n$

$$\frac{1}{n} \sum_{i=1}^{n} \frac{w\Big(f(k', \gamma_i)\Big) \cancel{\phi(\gamma_i)}}{\cancel{\phi(\gamma_i)}} \xrightarrow{p} \int w\Big(f(k', \gamma)\Big) \phi(\gamma) d\gamma$$

- Better approach: Gaussian Quadrature (for normally distributed shocks and from related families)

Optimization

- Find the minimum of some real-valued function of several real variables on a domain that has been specified.
  - Derivative methods: Newton's method, etc.
  - Derivative free: Golden section search, Grid search: pick maximizing node.
- Finding the global minimum can be challenging.
  - The function can have many local minima.
  - Curse of dimensionality & curvature of problem (when problems approach boundaries).

Conclusion

- Alternative: guess and verify (method of undetermined coefficients).
- We will cover this next time.
- Midterm after Spring Break (3/27).
- HW3 due in one week (3/6).