Quantitative Macro-Labor: Global Solution Techniques

Professor Griffy

UAlbany

Fall 2024

Announcements

- Today: value function iteration.
- Using:
  1. Grid search;
  2. Interpolation (grid search with functions filling in between nodes).
- Go through examples with MP94 model (code on cluster).
- Empirical regularities project due 11/5!

Solving a Model

- When we say "solve a model" what do we mean?
  1. Find the equilibrium of the model.
  2. Generally, determine the policy functions.
  3. Determine the transition equations given the individual and aggregate state.
  4. i.e., aggregate up the policy functions and determine prices given distributions.
- Generically, this is hard: many states, non-linear decision rules, etc.

Solving a Model

- Generically, this is hard: many states, non-linear decision rules, etc.
- Much of quantitative macro is about finding "shortcuts" without sacrificing accuracy of solution (some we have seen):
  1. Planner's problem: use welfare theorems to remove prices from problem.
  2. Rational expectations & complete markets: Aggregate worker decision rules by assuming they make same predictions about future prices, and face same consumption risk.
  3. Exogenous wage distribution/prices: agents do not respond to decisions of other agents.
  4. Block Recursive Equilibrium: agents face an equilibrium with individual prices, i.e., no need to know distribution.
- Linearization: assume the economy is close enough to steady-state that transition equations (i.e., policy functions) are close to linear within small deviations.
- Value function iteration: discretize state space and solve model at "nodes" in state space.

Discrete Mortensen and Pissarides (1994) Model

- iid productivity: draw $\epsilon \sim_{iid} F(\epsilon)$; evolve at rate λ
- Wages determined by Nash Bargaining (bargaining power α).
- agg shocks Z, endogenous separations when $\epsilon < \epsilon_d$
- Value of unemployment:

$$U(z) = b + \beta[p(\theta) \int_{\underline{\epsilon}}^{\bar{\epsilon}} [\max\{W(x, z'), U(z')\}] dF(x) + (1 - p(\theta)) U(z')]$$

- Value of employment:

$$W(\epsilon, z) = w + \beta E[\lambda\alpha \int_{\underline{\epsilon}}^{\bar{\epsilon}} [\max\{S(x, z'), 0\} - S(\epsilon, z')] \, dF(x) \\ + (1 - \lambda) W(\epsilon, z')]$$

- $S(x, z)$: joint surplus of firm & worker.

Firms

- Post vacancy at cost κ.
- Value of a filled vacancy:

$$J(\epsilon, z) = e^z \epsilon - w + \beta E[\lambda(1-\alpha) \int_{\underline{\epsilon}}^{\bar{\epsilon}} \max\{S(x, z'), 0\}\, dF(x)$$
$$+ (1-\lambda) J(\epsilon, z')]$$

- Value of unfilled vacancy:

$$V(z) = -\kappa + \beta E[q(\theta) \int_{\underline{\epsilon}}^{\bar{\epsilon}} [\max\{J(x, z'), V(z')\}] dF(x)$$
$$+ (1 - q(\theta)) V(z')]$$

- Free entry ($V = 0$)$\rightarrow$ match rate: $q(\theta) = \frac{\kappa}{\beta E[\int_{\epsilon_d} J(x,z') dF(x)]}$
- Market tightness: $\theta = q^{-1}(\frac{\kappa}{\beta E[\int J dF(x)]})$

Surplus and Employment Thresholds

- Impose matching func: $p(\theta) = A\theta^{1-\eta}$
- Surplus $S(\epsilon, z) = W(\epsilon, z) - U(z) + J(\epsilon, z) - V(z)$:

$$S(\epsilon, z) = e^z\epsilon - b + \beta\alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z')dF(x) + (1-\lambda)max\{S(\epsilon, z),$$
$$- A\theta^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z')dF(x)]$$
$$z' = \rho z + \epsilon_z,\ \epsilon_z \sim N(0, \sigma_\epsilon)$$

- How are we going to solve this model?
- Everything function of surplus.
- Set up grid of ϵ and z.

Value Function Iteration

- Basic approach to value function iteration:
  1. Create grid of points for each dimension in state-space.
  2. Specify terminal condition S_t for $t = T$ at each point in state-space.
  3. Solve problem of agent in period $T - 1$: $S_t(\epsilon, z) = e^z\epsilon - b + \beta E[func(\epsilon_d)]$.
  4. $\epsilon_d(z)$ is policy function, which yields the point where $S_t(\epsilon_d, z) = 0$
  5. Check to see if function has converged, i.e., $|S_t - S_{t+1}| < errtol \forall(\epsilon, z)$
  6. Update $S_{t+1} = S_t$
- Interpolation: same idea, but functions used to fill in between grid points.

Grids

- Want: smallest grids reasonable.
- Grids are both shocks, pick set number of standard deviations.
- Approximate a continuous AR(1) process with a markov process:
- Create grid of potential z values $\{z_1, ..., z_N\}$, approximate AR(1) process through transition probabilities.

$$E[z_t] = E[\rho z_{t-1} + \epsilon_{z,t}] = 0 \tag{1}$$

$$V[z_t] = V[\rho z_{t-1} + \epsilon_{z,t}] = \rho^2 \sigma_z^2 + \sigma_{\epsilon_z}^2 \tag{2}$$

$$\rightarrow (1 - \rho^2)\sigma_z^2 = \sigma_{\epsilon_z}^2 \tag{3}$$

- Define this process $G(\bar{\epsilon_z})$
- Tauchen (1986):

$$z_N = m\left(\frac{\sigma_{\epsilon_z}^2}{1 - \rho^2}\right) \tag{4}$$

$$z_1 = -z_N \tag{5}$$

$$z_2, ..., z_{N-1} \text{ equidistant} \tag{6}$$

Expectations with AR(1) Process

- Tauchen (1986):

$$z_N = m\left(\frac{\sigma^2_{\epsilon_z}}{1-\rho^2}\right) \tag{7}$$

$$z_1 = -z_N \tag{8}$$

$$z_2, ..., z_{N-1} \text{ equidistant} \tag{9}$$

- Create an *nxn* transition matrix Π using probabilities

$$\pi_{ij} = G(z_j + d/2 - \rho z_i) - G(z_j - d/2 - \rho z_i) \tag{10}$$

$$\pi_{i1} = G(z_1 + d/2 - \rho z_i) \tag{11}$$

$$\pi_{iN} = 1 - G(z_N + d/2 - \rho z_i) \tag{12}$$

- Idiosyncratic shocks (ϵ_z):
  - Right way to do it: Gaussian Hermite Quadrature.
  - Here: Same as above, set $\rho = 0$.

Endogenous Separations

- Problem:

$$S(\epsilon, z) = e^z \epsilon - b + \beta \alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z') dF(x) \\ + (1-\lambda) max\{S(\epsilon, z'), 0\} - A\theta^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z') dF(x)]$$

$$ln(z') = \rho ln(z) + \epsilon_z, \ \epsilon_z \sim N(0, \sigma_\epsilon)$$

- Find $\epsilon_d(z)$ such that $S(\epsilon_d, z) = 0$
- $S_0 =$? Safest bet to set it to zero at all ϵ, z.
- $\theta_0 =$? Safest bet to set it to zero at all ϵ, z.

Value Function First Iteration

- Intuitively, solve for surplus, find ϵ at which would like to separate for every z.
- Calculate the following:

$$e^z \epsilon_1 - b + \beta \times 0 \tag{13}$$
$$e^z \epsilon_2 - b + \beta \times 0 \tag{14}$$
$$\ldots \tag{15}$$
$$e^z \epsilon_N - b + \beta \times 0 \tag{16}$$

- Find ϵ_i st $S(\epsilon_i, z) = 0$.
- Repeat for all z.

Value Function First Iteration

- Now, check if problem has converged.
- What does this mean?
- The value in the current state is not changing over time.
- i.e., $S_t(\epsilon, z) \approx S_{t+1}(\epsilon, z)$.
- First iteration: it won't be.
- What do we do now?
- Update the continuation value:
- $S_{t+1} = S_t$ for all ϵ, z
- $\theta = q^{-1}(\frac{\kappa}{(1-\alpha)S})$
- Solve same problem again.

Value Function Second Iteration

- Solved for $S(\epsilon, Z)$ in previous iteration.
- Repeat, solving $S \; \forall \epsilon, \; z$

$$S(\epsilon, z) = e^z \epsilon - b + \beta \alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z') dF(x)$$
$$+ (1 - \lambda) max\{S(\epsilon, z'), 0\} - A\theta^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z') dF(x)]$$
$$ln(z') = \rho ln(z) + \epsilon_z, \; \epsilon_z \sim N(0, \sigma_\epsilon)$$

- Note that the continuation value is **not** zero!

$$e^z \epsilon_1 - b + \beta \times \text{Cont. Val} \tag{17}$$
$$e^z \epsilon_2 - b + \beta \times \text{Cont. Val} \tag{18}$$
$$\ldots \tag{19}$$
$$e^z \epsilon_N - b + \beta \times \text{Cont. Val} \tag{20}$$

Value Function Second Iteration

- We check again to see if it has converged.
- is $S_t(\epsilon, z) \approx S_{t+1}(\epsilon, z)$.
- What do we do now?
- Update the continuation value:
- $S_{t+1} = S_t$ for all ϵ, z
- $\theta = q^{-1}(\frac{\kappa}{(1-\alpha)S})$
- Solve same problem again.
- Keep doing this until the difference is very small.

Great, we're done!

- Not so fast: this isn't very accurate.
- Very slow if we have large numbers of states & grid points (scales exponentially).

Fundamental Problem

- The reason we need to use a computer to solve this problem is that we *don't know* the function $S(\epsilon, z)$.

$$S(\epsilon, z) = e^z\epsilon - b + \beta\alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z')dF(x)$$
$$+ (1-\lambda)max\{S(\epsilon, z'), 0\} - A\theta^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(x, z')dF(x)]$$
$$ln(z') = \rho ln(z) + \epsilon_z,\ \epsilon_z \sim N(0, \sigma_\epsilon)$$

- What is we *approximate* $S(\epsilon, z)$ with other functions?
- Some useful properties we can pick these functions to have:
  - Continuous
  - Differentiable
- If our approximation is accurate enough, we can drop some grid points!

Interpolation

- Call interpolated function $\hat{V}(k)$. Then,

$$S(\epsilon, z) = e^z\epsilon - b + \beta\alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'})dF(x)$$
$$+ (1-\lambda)max\{S(\hat{\epsilon, z'}), 0\} - A\hat{\theta}^{1-\eta}\int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'})dF(x)]$$
$$ln(z') = \rho ln(z) + \epsilon_z,\ \epsilon_z \sim N(0, \sigma_\epsilon)$$

- Where k' solves

$$e^z\epsilon - b + Cont\hat{.}\ Val = 0 \tag{21}$$

Updating

- We do *exactly the same thing as before*:

$$S(\epsilon, z) = e^z\epsilon - b + \beta\alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'})dF(x) \\ + (1-\lambda)max\{S(\hat{\epsilon, z'}), 0\} - A\hat{\theta}^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'})dF(x)] \tag{22}$$

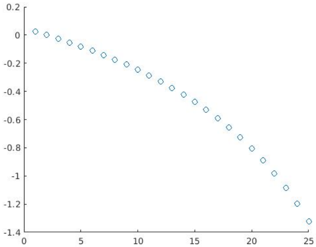
- For each z. Then, we check the convergence criteria:

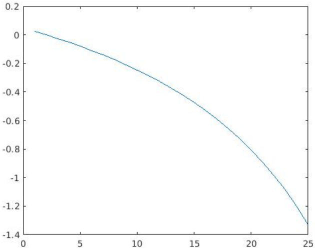
$$|S_t - S_{t+1}| < errtol \tag{23}$$

- How do we create the function $\hat{S}(\epsilon, z)$?
- "Connect the dots" of $S_t(\epsilon, z)$ between all ϵ levels in order for each z.
- In principle, interpolate in both dimensions, ϵ and z

Interpolation

- Left is function evaluated at sample points $x_1, ..., x_N$. Right is for (linearly) interpolated function:



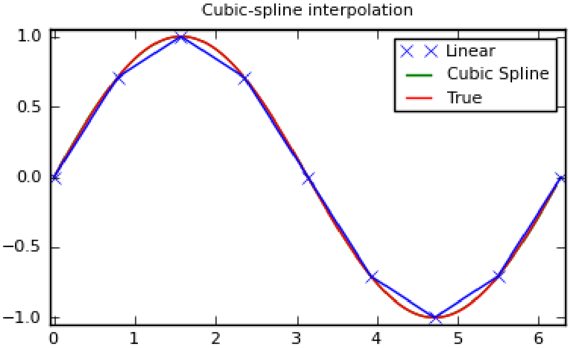


Interpolation

- In constructing our function $\hat{S}(\epsilon, z)$, we need to choose an interpolation scheme.
- Roughly, what *order* function do we believe will be accurate enough to mimick the value function:
  - First-order (linear)
  - Third-order (cubic)
  - Fifth-order (quintic)
- Some other useful interpolation routines:
  - PCHIP (piecewise cubic hermite interpolating polynomial): shape-preserving (not "wiggly") continuous 3rd order spline with continuous first derivative.

Interpolation

- Choice DOES matter:



Polynomial Interpolation

- Suppose we have a function $y = f(x)$ for which we know the values of y at $\{x_1, ..., x_n\}$.

- Then, the nth-order polynomial approximation to this function f is given by

$$f(x) \approx P_n(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \qquad (24)$$

- Then, we have a linear system with n coefficients.

- We could write this as $y = X\beta$. Look familiar?

Polynomial Interpolation

- We solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \tag{25}$$

- For $a_0, ..., a_n$

- What's the example we are all familiar with? Linear regression: $y = \alpha + X\beta$.

- In practice, this is computationally expensive, but this is the intuition.

Great, we're done!

- Not so fast: how do we handle expected values?
- Depends on expectation.
- Need an accurate way to perform numerical integration.

Surplus function

- Problem:

$$
\begin{aligned}
S(\epsilon, z) = e^{z}\epsilon - b + \beta\alpha E[\lambda \int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'}) dF(x) \\
+ (1-\lambda) max\{S(\hat{\epsilon, z'}), 0\} - A\hat{\theta}^{1-\eta} \int_{\epsilon_d}^{\bar{\epsilon}} S(\hat{x, z'}) dF(x)] \\
ln(z') = \rho ln(z) + \epsilon_z,\ \epsilon_z \sim N(0, \sigma_\epsilon)
\end{aligned}
$$

- Make sure your process for z stays non-negative.

Expectations Generally

- Expected values also need to be calculated carefully.
- Continuation surplus from before:

$$E[V(\epsilon, z')] \tag{26}$$

- If *not* an AR(1)/markov process, need to approximate integral.
- Generically, pick function f and weights w_i

$$E[V(\epsilon, z')] = \int_a^b f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i) \tag{27}$$

- x_i may be known or picked optimally.
- We will return to this in the future.

Next Time

- Intro to heterogeneous agents.
- Empirical regularities project due soon!