Quantitative Macro-Labor: Solving through Linearizing

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Announcements

- Today: Start discussing solution techniques.
- Focus on linearization.
- Discuss the upsides and downsides.
- Next time: global techniques.
- Empirical regularities project is due in two weeks!

Solving a Model

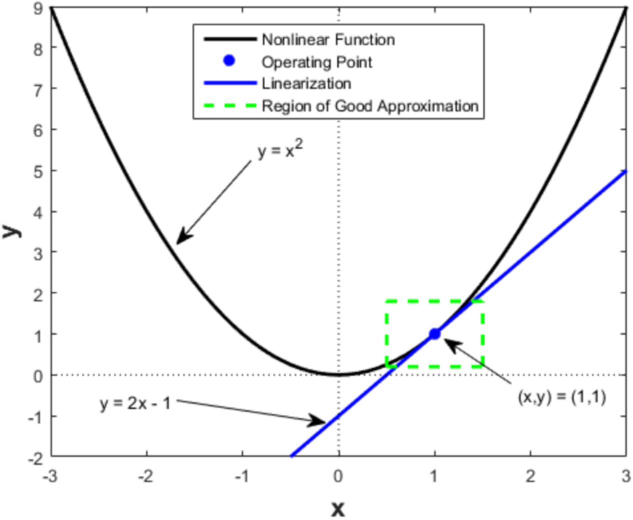
- When we say "solve a model" what do we mean?
    1. Find the equilibrium of the model.
    2. Generally, determine the policy functions.
    3. Determine the transition equations given the individual and aggregate state.
    4. i.e., aggregate up the policy functions and determine prices given distributions.
- Generically, this is hard: many states, non-linear decision rules, etc.

Solving a Model

- Generically, this is hard: many states, non-linear decision rules, etc.
- Much of quantitative macro is about finding "shortcuts" without sacrificing accuracy of solution (some we have seen):
  1. Planner's problem: use welfare theorems to remove prices from problem.
  2. Rational expectations & complete markets: Aggregate worker decision rules by assuming they make same predictions about future prices, and face same consumption risk.
  3. Exogenous wage distribution/prices: agents do not respond to decisions of other agents.
  4. Block Recursive Equilibrium: agents face an equilibrium with individual prices, i.e., no need to know distribution.
- Linearization: assume the economy is close enough to steady-state that transition equations (i.e., policy functions) are close to linear within small deviations.

Linearization - Intuition

- Approximate a function around point where second derivative small (in abs. terms)
- Our models: around steady-state.



Linearization - Basic Approach

- Write model to incorporate out of steady-state equilibria in expectations.
- If not: just comparative statics.
- Select point around which changes in dynamics are small.
- Use Taylor Series approximation evaluated at point.
- Taylor Series:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... \quad (1)$$

Linearization - Example

- Taylor Series:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... \tag{2}$$

- Ex: approximate function $f(x) = ln(1+r)$ for small r.

$$\begin{aligned} f(x) &= ln(1+r) && (3) \\ &\approx ln(1) + \frac{1}{1}((1+r)-1) && (4) \\ &\approx r && (5) \\ & && (6) \end{aligned}$$

- This is why we coefficients in regressions are interpreted as percent change when LHS is in logs.

Local Linear Methods

- Log-linearize the system around the steady-state, then proceed.
- In general, must determine linearized equilibrium, then apply solution algorithm.
- Solution algorithms:
  1. Klein's Method (2000): Used for singular matrices.
  2. Sim's Method (2001): Used when it is unclear which variables are states and controls.
  3. Blanchard and Kahn's Method (1980): First solution method for rational expectations models.
- Basic idea of these techniques: find $x(0)$ such that system arrives at steady-state.

The DMP Model ("Ch. 1 of Pissarides (2000)")

- Agents:
  1. Employed workers;
  2. unemployed workers;
  3. vacant firms;
  4. matched firms.
- Linear utility ($u = b, u = w$) and production $y = z > b$.
- Matching function:
  1. Determines *number* of meetings between firms & workers.
  2. Args: levels searchers & vacancies ($U = u \times L, V = v \times L$)
  3. Constant returns to scale (L is lab. force):

$$M(uL, vL) = uL \times M(1, \frac{v}{u}) = uL \times p(\theta) \tag{7}$$

  4. where $\theta = \frac{v}{u}$ is "labor market tightness"
  5. Match rates:

$$\underbrace{p(\theta)}_{Worker} = \theta \underbrace{q(\theta)}_{Firm} \tag{8}$$

Value Functions

- Unemployed flow value:

$$rU = b + p(\theta)E[W(w) - U] \tag{9}$$

- Employed flow value:

$$rW(w) = w + \delta[U - W(w)] \tag{10}$$

- Vacant flow value:

$$rV = -\kappa + q(\theta)E[J(w) - V] \tag{11}$$

- Matched flow value:

$$rJ(w) = (p - w) + \delta[V - J(w)] \tag{12}$$

- Free entry equilibrium condition:

$$rV = 0 \tag{13}$$

$$\rightarrow \frac{\kappa}{E[J(w)]} = q(\theta) \tag{14}$$

Equilibrium

- The equilibrium we have described is a steady-state equilibrium characterized by value functions U, W, J, V, a wage function w, a market tightness function θ, and steady-state level unemployment u, such that
  1. A steady-state level of unemployment, derived from the flow unemployment equation.
  2. A wage rule that splits the surplus of a match according to a sharing rule with bargaining weight β
  3. A free entry condition that determines θ given wages and steady-state unemployment.

Equilibrium Objects

- Three key equilibrium objects:
  1. Wages;
  2. unemployment;
  3. $\theta = \frac{v}{u}$ (vacancies).
- How do we solve for each if we can't assume the model is in the steady-state?
- i.e., what happens when model economy is in *transition*?
- Policy functions (w, θ) are forward-looking: must re-solve model to incorporate dynamics.
- Intuition: high or low productivity periods may change asset value of jobs.

Worker Value Functions

- Value functions:
  1. Employed at wage w: $W(w)$
  2. Unemployed: U.
- Unemployed flow value:

$$rU = b + p(\theta)[W(w) - U] + \dot{U} \tag{15}$$

- Employed flow value:

$$rW(w) = w + \delta[U - W(w)] + \dot{W}(w) \tag{16}$$

Firm Value Functions

- Value functions:
  1. Filled, paying wage w: $J(w)$
  2. Vacant V.
- Vacant flow value:

$$rV = -\kappa + q(\theta)[J(w) - V] + \dot{V} \tag{17}$$

- Matched flow value:

$$rJ(w) = (z - w) + \delta[V - J(w)] + \dot{J(w)} \tag{18}$$

Surplus

- I emphasized in endogenous separation lecture, but surplus again key.
- Search and matching models with Nash Bargaining: surplus determines all policy functions.
- Surplus:

$$S(w) = W(w) + J(w) - U - V \tag{19}$$

$$\dot{S(w)} = \dot{W(w)} + \dot{J(w)} - \dot{U} - \dot{V} \tag{20}$$

- Similarly, dynamics of surplus equation will determine all dynamics of model.

Assumptions

- To solve this model, we will make two assumptions about dynamics:
  1. Free entry **always** holds;
  2. Wages are continuously renegotiated.
- Meaning:

$$V = 0, \dot{V} = 0 \tag{21}$$
$$w = \beta S(w) \tag{22}$$
$$\dot{w} = \beta \dot{S}(w) \tag{23}$$

- If these were only two equilibrium objects: aggregate shocks would immediately move model to new steady-state.
- But unemployment is **not** a jump variable.

Dynamics

- Know from free entry that

$$q(\theta) = \frac{\kappa}{J(w)} \tag{24}$$

$$J = (1-\beta)S \tag{25}$$

$$\rightarrow S = \frac{\kappa}{(1-\beta)q(\theta)} \tag{26}$$

$$\dot{S} = -\frac{\kappa q'(\theta)\dot{\theta}}{(1-\beta)(q(\theta))^2} \tag{27}$$

- Surplus:

$$(r+\delta)S = z - b + p(\theta)\beta S + \dot{S} \tag{28}$$

$$\rightarrow 0 = \frac{\kappa q'(\theta)\dot{\theta}}{(q(\theta))^2} + \frac{\kappa(r+\delta+p(\theta)\beta)}{q(\theta)} - (1-\beta)(z-b) \tag{29}$$

Dynamics II

- Wages continuously renegotiated, thus change with surplus.
- Only equilibrium object that does not jump: unemployment.
- Recall flow equation for unemployment:

$$\dot{u} = \delta(1-u) - p(\theta)u \tag{30}$$

- By free entry condition, we know that $\theta = \theta^*$. Vacancies adjust to new steady-state ratio.

$$\rightarrow \dot{u} = \delta(1-u) - p(\theta^*)u \tag{31}$$

- $\dot{u} \neq 0$? Because firms may hire more or less of unemployed pool.

Linearization

- Thus far, we haven't taken a stand on our solution technique.
- We have equations that determine out of steady-state dynamics for
  1. Surplus: θ and wages;
  2. Unemployment.
- Rest of class: use this as jumping off point to solve linearized version of model.
- What do we need to linearize? Surplus equation.
- Approach:
  1. Compute steady-state.
  2. Linearize around steady-state.
  3. Solve linear system.

Linearization II

- Steady-state θ:

$$0 = \frac{\kappa q'(\theta)\dot{\theta}}{(q(\theta))^2} + \frac{\kappa(r+\delta+p(\theta)\beta)}{q(\theta)} - (1-\beta)(z-b) \quad (32)$$

$$\rightarrow (1-\beta)(z-b) = \frac{\kappa(r+\delta+\theta q(\theta^*)\beta)}{q(\theta^*)} \quad (33)$$

- From here:
- Solve for $\dot{\theta} = f(\theta)$.
- Linearize:

$$f(\theta) \approx f(\theta^*) + f'(\theta^*)(\theta - \theta^*) \quad (34)$$

Linearization III

- Steady-state θ:

$$0 = \frac{\kappa q'(\theta)\dot{\theta}}{(q(\theta))^2} + \frac{\kappa(r + \delta + p(\theta)\beta)}{q(\theta)} - (1-\beta)(z-b) \tag{35}$$

- Solving for $\dot{\theta} = f(\theta)$:

$$\dot{\theta} = \frac{\beta q(\theta)^2}{q'(\theta)}(\frac{1-\beta}{\beta}\frac{z-b}{\kappa} - \frac{r+\delta}{\beta q(\theta)} - \theta) \tag{36}$$

- Taylor approximation of $\dot{\theta}$ around θ^*:

$$\dot{\theta} \approx \cancel{f(\theta^*)} + [(r+\delta) - \frac{\beta q(\theta^*)^2}{q'(\theta^*)}](\theta - \theta^*) \tag{37}$$

$$\dot{\theta} \approx A\Delta\theta \tag{38}$$

- What does $A > 0$ mean?

Stability

- Taylor approximation of $\dot{\theta}$ around θ^*:

$$\dot{\theta} \approx \cancel{f(\theta^*)} + [(r+\delta) - \frac{\beta q(\theta^*)^2}{q'(\theta^*)}](\theta - \theta^*) \tag{39}$$

$$\dot{\theta} \approx A\Delta\theta \tag{40}$$

- If $A > 0$, no solution.
- Here: must be that $\dot{\theta} = 0$.

Unemployment

- Only dynamics: unemployment.

$$\dot{u} = \delta(1 - u) - p(\theta^*)u \tag{41}$$

- If $\theta_2^* << \theta_1^*$, $u \uparrow$.

Bad Example of Linearizing

- Ex-post: this wasn't the best model to explore linearization methods in.
- Euler Equation from Hansen (1985):

$$\frac{\eta}{c_t} = \beta E_t[\frac{1}{c_{t+1}}(\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)] \quad (42)$$

- Next steps: solve for steady-state.
- Here: show how to linearize this model

Linearized Euler Equation

- Define deviations from steady-state: $\tilde{x}_t = ln(x_t) - ln(x)$:

$$\begin{aligned}\tilde{x}_t &\approx \tilde{x}_t(x) + \frac{\partial \tilde{x}_t}{\partial x_t}(x)(x_t - x)\\ &\approx ln(1) + \frac{1}{x}(x_t - x)\\ &\approx x(1 + \tilde{x}_t)\end{aligned}$$

$$\begin{aligned}\frac{\eta}{c_t} &= \beta E_t[\frac{1}{c_{t+1}}(\theta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta)]\\ \frac{\eta}{ce^{\tilde{c}_t}} &= \beta E_t[\frac{1}{ce^{\tilde{c}_{t+1}}}(\theta(\frac{ye^{\tilde{y}_{t+1}}}{ke^{\tilde{k}_{t+1}}}) + 1 - \delta)]\\ \frac{\eta}{\beta}e^{-\tilde{c}_t} &= E[e^{-\tilde{c}_{t+1}}(\theta \frac{y}{k}e^{\tilde{y}_{t+1}}e^{-\tilde{k}_{t+1}} + 1 - \delta)]\\ \rightarrow \frac{\eta}{\beta}(1 - \tilde{c}_t) &= E[(1 - \tilde{c}_{t+1})(\theta \frac{y}{k}(1 + \tilde{y}_{t+1})(1 - \tilde{k}_{t+1}) + 1 - \delta)]\end{aligned}$$

- Using $e^{-\tilde{x}_t} \approxeq (1 - \tilde{x}_t)$.

Linearized Euler Equation II

- Percent change X percent change ≈ 0 ($\tilde{x}_t \times \tilde{y}_t \approx 0$)

$$\frac{\eta}{\beta}(1-\tilde{c}_t) = E[(1-\tilde{c}_{t+1})(\theta\frac{y}{k}(1+\tilde{y}_{t+1}-\tilde{k}_{t+1})+1-\delta)]$$

$$\frac{\eta}{\beta}(1-\tilde{c}_t) = E[\theta\frac{y}{k}(1+\tilde{y}_{t+1}-\tilde{k}_{t+1})+1-\delta - \tilde{c}_{t+1}\theta\frac{y}{k} - \tilde{c}_{t+1} + \delta\tilde{c}_{t+1}] \quad (43)$$

- Steady-state value of capital given by

$$k^* = (\frac{\theta}{\frac{\eta}{\beta}-1+\delta})y^* \quad (44)$$

Linearized Euler Equation III

- Percent change X percent change ≈ 0 ($\tilde{x}_t \times \tilde{y}_t \approx 0$)

$$\frac{\eta}{\beta}(1-\tilde{c}_t) = E[\theta \frac{y}{(\frac{\theta}{\frac{\eta}{\beta}-1+\delta})y}(1+\tilde{y}_{t+1}-\tilde{k}_{t+1}) + 1 - \delta - \tilde{c}_{t+1}\theta \frac{y}{(\frac{\theta}{\frac{\eta}{\beta}-1+\delta})y} - \tilde{c}_{t+1} + \delta\tilde{c}_{t+1}] \quad (45)$$

$$\frac{\eta}{\beta}(1-\tilde{c}_t) = E[(\frac{\eta}{\beta}-1+\delta)(1+\tilde{y}_{t+1}-\tilde{k}_{t+1}) + 1 - \delta - (\frac{\eta}{\beta}-1+\delta)\tilde{c}_{t+1} - \tilde{c}_{t+1} + \delta\tilde{c}_{t+1}] \quad (46)$$

- Yielding the following linearized Euler Equation:

$$0 = \frac{\eta}{\beta}\tilde{c}_t + E[(\frac{\eta}{\beta}-1+\delta)(\tilde{y}_{t+1}-\tilde{k}_{t+1}) - \frac{\eta}{\beta}\tilde{c}_{t+1}] \quad (47)$$

- This is precisely the form we want: $0 = f(x)$.

Log-Linearizing the System

- Static variables: output (y_t), hours (h_t), investment (i_t).
- Dynamic variables: capital (k_t) and consumption (c_t).
- Stochastic dynamic variable: productivity (a_t).
- We can now write the system as:

$$\Psi_1 \zeta_t = \Psi_2 \xi_t + \Psi_3 \tilde{a}_t \tag{48}$$

$$\Psi_4 E_t(\xi_{t+1}) = \Psi_5 \xi_t + \Psi_6 \zeta_t + \Psi_7 \tilde{a}_t \tag{49}$$

- ζ_t are static predetermined and nonpredetermined variables, $[\tilde{y}_t, \tilde{h}_t, \tilde{i}_t]'$.
- ξ_t are dynamic predetermined and nonpredetermined variables, $[\tilde{k}_t, \tilde{c}_t]'$.
- $\tilde{a}_t$ is the technology process.
- Why is $\tilde{c}_t$ among the dynamic variables?

Next Time

- Global Solutions to the DMP model.
- Empirical regularities project due very soon!